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On the Regularity Condition for Vector Programming Problems

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Abstract. In this work, we use a notion of approximation derived from Jourani and Thibault [13] to ascertain optimality conditions analogous to those that established but applicable to larger class of vector valued objective mappings and constraint set-valued mappings. To this end, we introduce an appropriate regularity condition to help us discern the Karush-Kuhn-Tucker multipliers.

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1. Introduction

A lot of research has been carried out on multiobjective optimization problems [2, 3, 7, 9, 15, 16]. Corley [7] has given the optimality conditions for convex and nonvonvex multiobjective problems in terms of the Clarke derivative. Luc [15] also provides the optimality conditions for data that are upper semidifferentiable. Luc and Malivert [16] extend the concept of invex functions to invex multifunctions and study the optimality conditions for multiobjective optimization with invex data in terms of the contingent derivative.

In this paper, we are concerned with the generalized optimization problem

$$(P) : \begin{cases} f(x) \to M - \text{optimal} \\ \text{subject to } 0 \in F(x) \end{cases}$$

where f is a mapping of a Banach space X onto the n-dimensional Euclidean space $\mathbb{R}^n$, $M \subset \mathbb{R}^n$ is a nonempty convex cone with $M \neq -M$, and F is a set-valued mapping of X onto another Banach space Y.

In Dien [9], the optimization problem (P) was studied when the data f and F are locally Lipschitz and X, Y are Hilbert spaces. In this note, we somewhat extend Dien's findings by seeing if they are valid for larger class of problems with objective mappings f (respectively, set valued mappings F) admitting approximations (respectively, whose support functions admit approximations). Also here, X and Y will be Banach spaces.

Our approach consists of using the notion of approximation which is introduced for the first time by Jourani and Thibault [13] and revised after by Allali and

Amahroq [1]. Here we adopt the latest definition of approximation [1] to detect an appropriate regularity condition and Karush-Kuhn-Tucker multipliers. An example of a non Lipschitz function that allows an approximation is given in Proposition 2.1. In addition, the regularity condition used in this paper is more general than that in [9], since functions that are locally Lipschitzian admit the Clarke's subdifferential as approximations.

The outline of the paper is as follows: preliminary results are described in Section 2; the main result is given in Section 3; Sections 4 discusses an application to a mathematical programming problem.

2. Preliminaries

Let X and Y be Banach spaces. We denote by $L(X, Y)$ the set of continuous linear mappings between X and Y, $\mathbb{B}_Y$ the closed unit ball of Y centered at the origin, $\mathbb{S}_Y$ the unit sphere of Y and X^* the continuous dual of X. We write $\langle ., . \rangle$ for the canonical bilinear form with respect to the duality $\langle X^*, X \rangle$.

Let C be the set of all $x \in X$ satisfying $0 \in F(x)$ and let $M \subset \mathbb{R}^n$ be a nonempty convex cone with $M \neq -M$. Following [9], a point $\overline{x} \in C$ is said to be an $M-$optimal solution for problem (P) if for any point $x \in C$ satisfying $f(x) - f(\overline{x}) \in M$ one has $f(\overline{x}) - f(x) \in M$ which implies that $f(x) = f(\overline{x})$ whenever M is pointed. In fact, this $M-$optimal solution is a global solution for the optimization problem (P).

By M^* we denote the polar cone of M

$$M^* := \left\{ m^* \in \mathbb{R}^n : < m^*, m > \leqslant 0 \;\; \forall m \in M \right\}.$$

For every $y^* \in Y^*$ the support function of F at x is defined as follows:

$$C_F(y^*, x) := \sup_{y \in F(x)} < y^*, y > .$$

We suppose that the barrier cone of F

$$Y_F^* := \left\{ y^* \in Y^* : \; \sup_{y \in F(x)} < y^*, y > < +\infty \right\}$$

is closed and does not depend on x. For example, this is the case when F is locally Lipschitz. The distance function of F to zero,

$$d(0, F(x)) = \inf \{ \|y\| : y \in F(x) \}$$

is related to the support function of F by the relation

$$d(0, F(x)) = \max_{y^* \in Y_F^* \cap \mathbb{B}_{Y^*}} -C_F(y^*, x).$$

If $d(0, F(x)) > 0$ then there is a unique $y^* \in Y_F^* \cap \mathbb{B}_{Y^*}$ satisfying $\|y^*\| = 1$ and $d(0, F(x)) = -C_F(y^*, x)$, see [8] and [19].

DEFINITION 1. [1]. *Let f be a mapping from X into Y, $\overline{x} \in X$ and $A_f(\overline{x}) \subset L(X, Y)$. $A_f(\overline{x})$ is said to be an approximation of f at $\overline{x}$ if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$f(x) - f(\overline{x}) \in A_f(\overline{x})(x - \overline{x}) + \varepsilon\|x - \overline{x}\|\mathbb{B}_Y \tag{1}$$

for all $x \in \overline{x} + \delta\mathbb{B}_X$.

It is easy to see that $f+g$ has $A_f(\overline{x})+A_g(\overline{x})$ as an approximation at $\overline{x}$ whenever $A_f(\overline{x})$ and $A_g(\overline{x})$ are approximations of f and g at $\overline{x}$.

Note that $A_f(\overline{x})$ is a singleton if and only if f is Fréchet differentiable at $\overline{x}$. In [1], it is shown that when f denotes a function that is locally Lipschitzian it admits as an approximation the Clarke subdifferential of f at $\overline{x}$; i.e.

$$A_f(\overline{x}) = \partial f(\overline{x}) := co\{\text{Lim}\, \nabla f(x_n); x_n \in \text{dom}\nabla f \text{ and } x_n \to \overline{x}\}.$$

In order to give an example of a non locally Lipschitz function, let us recall the following definition.

DEFINITION 2. [17]. *Let $f : X \to \overline{R} := [-\infty, +\infty]$ be an extended-real-valued function and $\overline{x} \in dom(f)$. The symmetric subdifferential of f at $\overline{x}$ is defined by*

$$\partial^0 f(\overline{x}) := \partial f(\overline{x}) \sqcup \left[-\partial(-f)(\overline{x})\right]$$

where $\partial f(\overline{x}) := lim\, sup_{x\to^f\overline{x},\varepsilon\searrow 0}\widehat{\partial}_\varepsilon f(x)$ and $\widehat{\partial}_\varepsilon f(x)$ is the $\varepsilon-$ Fréchet subdifferential of f at x. For more details see [17].

Note that sufficient conditions for the upper semicontinuity of $\partial^0 f(.)$ can be found in [12] and [14].

PROPOSITION 1. *Let $f : \mathbb{R}^p \to \overline{R} := [-\infty, +\infty]$ be continuous and $\overline{x} \in dom(f)$. Then $\partial^0 f(\overline{x})$ is an approximation of f at $\overline{x}$.*

Proof. The proof is similar to that of [1, Proposition 2.1.2] by using the mean value theorem [17, Theorem 5.7]. □

LEMMA 1. *Let $x^* \in \mathbb{B}_{\mathbb{R}^n}$ and let f be a mapping of X onto $\mathbb{R}^n$ which admits $A_f(\overline{x})$ as an approximation at $\overline{x}$. Then $x^* \circ A_f(\overline{x})$ is an approximation of $\langle x^*, f(.)\rangle$ at $\overline{x}$.*

Proof. It is obvious. □

A chain rule calculus for Lipschitz mappings has been established by Jourani and Thibault in [13].Here we state a more general and inclusive situation since the point $\overline{x}$ in (1) varies in the definition of [13].

THEOREM 1. *Consider three Banach spaces X, Y and Z. Assume that $A_f(\overline{x})$ is a bounded approximation of $f : X \to Y$ at $\overline{x}$ and $A_g(\overline{y})$ is a bounded approximation of $g : Y \to Z$ at $\overline{y} = f(\overline{x})$. Then $A_g(\overline{y}) \circ A_f(\overline{x})$ is an approximation of $g \circ f$ at $\overline{x}$.*

Proof. Direct verifications completes the proof. □

The following result is an extension of [5, Proposition 2.3.12] obtained for Lipschitz functions.

COROLLARY 1. *Suppose for $i = 1, 2, ..., n$, the function f_i admits a bounded approximation $A_{f_i}(\overline{x})$ at $\overline{x}$. Let*

$$h(x) = \max\{f_i(x) : i = 1, 2, ..., n\}$$

and $I(\overline{x}) := \{i : f_i(\overline{x}) = h(\overline{x})\}$. Then $co\left\{A_{f_i}(\overline{x}) : i \in I(\overline{x})\right\}$ is an approximation of h at $\overline{x}$, where "co" denotes the convex hull.

Proof. The proof is a consequence of Theorem 1. □

Similarly, we deduce the following result which is a variant of [6, Theorem 2.8.2].

COROLLARY 2. *Suppose that $\mathbb{T}$ is separable, and $\{f_t\}_{t\in\mathbb{T}}$ is a collection of functions f_t which admit bounded approximations $A_{f_t}(\overline{x})$ at $\overline{x}$.*
Set $h(x) := sup_{t\in\mathbb{T}}\{f_t(x)\}$ and $J(\overline{x}) := \{t \in \mathbb{T} : f_t(\overline{x}) = h(\overline{x})\}$.
If $t \mapsto f_t(\overline{x})$ is upper semicontinuous, then $co\left\{A_{f_t}(\overline{x}) : t \in J(\overline{x})\right\}$ is an approximation of h at $\overline{x}$.

The next corollary is an extension of [8, Proposition 2.2].

COROLLARY 3. *Suppose that $\mathbb{B}_{Y^*}$ is separable. Then for all $x \in X$, the distance function of F admits*

$$co\left\{-A_{C_F(y^*,.)}(x) : y^* \in J(x)\right\}$$

as an approximation at x, where $J(x) = \left\{y^* \in Y_F^* : \|y^*\| \leqslant 1,\ d(0, F(x)) = -C_F(y^*, x)\right\}$.
If in addition, $d(0, F(x)) > 0$, then $J(x)$ consists of only one single element y^* with $\|y^*\| = 1$.

The definition that we propose below is more comprehensive than Dien's [9]; however, the two are identical when the data are Lipschitz and the Clarke's subdifferential is taken as the approximation.

DEFINITION 3. *The problem (P) is said to be regular at $\overline{x} \in C$ if there exist a neighborhood U of $\overline{x}$ and $\delta, \gamma > 0$ such that :*
$\forall y^* \in Y_F^*\ \forall x \in U\ \forall x^* \in A_{C_F(y^*,\cdot)}(x)\ \exists \xi \in \delta\mathbb{B}_X$ *such that*

$$C_F(y^*, x) + \langle x^*, \xi\rangle \geqslant \gamma\|y^*\|.$$

Note that, with appropriate data, Zowe and Kurcyusz's regularity [20] implies the above regularity.

In what follows, the function f and the set-valued mapping F are assumed to have the following properties:

(i) The set-valued mapping $(y^*, x) \to A_{C_F(y^*,\cdot)}(x)$ is upper semicontinuous when X^*, Y^* are endowed with the weak-star topology and X with the strong topology, that is, if $x_n^* \in A_{C_F(y_n^*,\cdot)}(x_n)$ where $x_n^* \overset{w^*}{\to} x^*$ in X^*, $y_n^* \overset{w^*}{\to} y^*$ in Y^* and $x_n \to x$ in X, then $x^* \in A_{C_F(y^*,\cdot)}(x)$.

(ii) There exists $\delta > 0$ such that for every $x \in \overline{x} + \delta\mathbb{B}_X$, f admits an approximation $A_f(x)$ at x and $A_f(\overline{x})$ is bounded w^*-closed.

(iii) There exists $\delta > 0$ such that $C_F(y^*, \cdot)$ admits an approximation $A_{C_F(y^*,.)}(x)$ at x and

$$\alpha A_{C_F(y^*,.)}(x) \subset A_{\alpha C_F(y^*,.)}(x)$$

for every $y^* \in Y_F^*$, $x \in \overline{x} + \delta\mathbb{B}_X$ and $\alpha > 0$.

(iv) For each $\varepsilon > 0$, f is ε-approximately upper semicontinuous at $\overline{x}$, that is, there exists a real number $\delta > 0$ such that for all $x \in \overline{x} + \delta\mathbb{B}_X$

$$A_f(x) \subset A_f(\overline{x}) + \varepsilon\mathbb{B}_{X^*}.$$

(v) For each $\varepsilon > 0$, the set-valued mapping F is ε-sequentially upper semicontinuous at $\overline{x}$. i.e. for all $(z_n^*) \to z^*$ in $\mathbb{S}_{Y^*}$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$A_{C_F(z_n^*,.)}(x) \subset A_{C_F(z^*,.)}(\overline{x}) + \varepsilon\mathbb{B}_{X^*}$$

for all $n \geqslant n_0$ and all $x \in \overline{x} + \delta\mathbb{B}_X$.

3. Optimality Conditions.

In all that follows, we suppose the separability of X and $\mathbb{B}_{Y^*}$ and that F is less semicontinuous at $(\overline{x}, 0) \in Gr(F)$.
Now we propose our main result where the argument is similar to that used by Dien in [9], but we give the proof in a more general situation.

THEOREM 2. *Suppose that $\overline{x}$ is a M-optimal solution of (P).*
If (P) is regular at $\overline{x}$, then there exist vectors $m^* \in M_1^* := M^* \cap \mathbb{S}_{Y^*}$, $y^* \in Y_F^*$ such that

$$\begin{cases} 0 \in m^* \circ A_f(\overline{x}) - A_{C_F(y^*,.)}(\overline{x}), \\ \qquad C_F(y^*, \overline{x}) = 0. \end{cases}$$

Proof. Since M is a convex cone and $M \neq -M$, then using a separation theorem, one can find $m_1 \in ri(M)$ such that $\|m_1\| = 1$ and $-m_1 \notin cl(M)$.
Define $M_n := \dfrac{1}{n}m_1 + cl(M)$; then M_n is a closed convex set contained in M and $0 \notin M_n$.

Consider the set-valued map E_n from X into $\mathbb{R}^n$ defined by

$$E_n(x) := f(\overline{x}) - f(x) + M_n.$$

Let $\Psi_1(x) := d(0, E_n(x))$, $\Psi_2(x) := d(0, F(x))$ and $h_n(x) := \max(\Psi_1(x), \Psi_2(x))$; we have $h_n(\overline{x}) \leqslant \frac{1}{n} + \inf_{x\in X} h_n(x)$.

By using Ekeland's Variational Principle [10], there exists $x_n \in X$ such that

$$\begin{cases} \|x_n - \overline{x}\| \leqslant \dfrac{1}{\sqrt{n}} \\ h_n(x_n) \leqslant h_n(x) + \dfrac{1}{\sqrt{n}} \|x - x_n\| \qquad \text{for all } x \in X. \end{cases}$$

Hence x_n is a minimum of $h_n(x) + \frac{1}{\sqrt{n}} \|x - x_n\|$. Using [1, Theorem 3.1.1] we get

$$0 \in cl^* co \left[A_{h_n}(x_n) + \frac{1}{\sqrt{n}} \mathbb{B}_{X^*} \right].$$

In view of Corollary 1, it follows that $co \left\{ A_{\Psi_i} : i \in I(x_n) \right\}$ is an approximation of h_n at x_n, where $I(x_n) := \{i : h_n(x_n) = \Psi_i(x_n)\}$.

Consequently, there exists $\lambda_n \in [0, 1]$ such that

$$0 \in \lambda_n A_{\Psi_1}(x_n) + (1 - \lambda_n) A_{\Psi_2}(x_n) + \frac{1}{\sqrt{n}} \mathbb{B}_{X^*}$$

where $\lambda_n = 0$ if $\Psi_1(x_n) < \Psi_2(x_n)$, $\lambda_n = 1$ if $\Psi_2(x_n) < \Psi_1(x_n)$, and $0 < \lambda_n < 1$ if $\Psi_1(x_n) = \Psi_2(x_n)$.

Moreover $\max(\Psi_1(x_n), \Psi_2(x_n)) > 0$, otherwise $d(0, E_n(x_n)) = d(0, F(x_n)) = 0$. So that $x_n \in C$ and $f(x_n) - f(\overline{x}) \in M_n \subset M$. On the other hand, $f(\overline{x}) - f(x_n) \in M$, since $\overline{x}$ is M-optimal. Consequently, $0 \in M_n + M \subset M_n$, a contradiction.

Using Corollary 3, there exist $m_n^* \in M^* \cap \mathbb{S}_{Y^*}$, $y_n^* \in Y_F^* \cap \mathbb{S}_{Y^*}$ and a real number $\lambda_n \in [0, 1]$ such that

$$0 \in \lambda_n \; A_{<m_n^*, f(.)>}(x_n) - (1 - \lambda_n) \; A_{C_F(y_n^*, .)}(x_n) + \frac{1}{\sqrt{n}} \mathbb{B}_{X^*}.$$

We deduce by Lemma 1 that

$$0 \in \lambda_n \; m_n^* \circ A_f(x_n) - (1 - \lambda_n) \; A_{C_F(y_n^*, .)}(x_n) + \frac{1}{\sqrt{n}} \mathbb{B}_{X^*}. \tag{2}$$

Taking a subsequence if necessary, we can assume that $(\lambda_n) \to \lambda \in [0, 1]$, $(m_n^*) \to m^* \in M^* \cap \mathbb{S}_{Y^*}$ and $(y_n^*) \overset{w^*}{\to} \widetilde{y}^* \in Y_F^* \cap \mathbb{B}_{Y^*}$, when n tends to $+\infty$.
We have $\lambda > 0$. Indeed, by (2) we can choose $x_{1n}^* \in m_n^* \circ A_f(x_n)$, $x_{2n}^* \in A_{C_F(y_n^*, .)}(x_n)$ and $x_{3n}^* \in \mathbb{B}_{X^*}$ such that

$$\lambda_n \; x_{1n}^* + \frac{1}{\sqrt{n}} \; x_{3n}^* = (1 - \lambda_n) \; x_{2n}^*. \tag{3}$$

Since (P) is regular at $\overline{x}$, for every n there exists $\xi_n \in \delta\mathbb{B}_X$ such that

$$C_F\left(y_n^*, x_n\right) + \left\langle x_{2n}^*, \xi_n\right\rangle \geqslant \gamma \tag{4}$$

Combining (3) and (4), it yields

$$\lambda_n \langle x_{1n}^*, \xi_n\rangle + \frac{1}{\sqrt{n}}\delta \geqslant (1-\lambda_n)\left[\gamma + d(0, F(x_n))\right]. \tag{5}$$

Since $\lim_{n\to\infty} d(0, F(x_n)) = d(0, F(\overline{x})) = 0$ and $\left\|x_{1n}^*\right\| \leqslant \alpha := \sup_{x^*\in A_f(\overline{x})} \|x^*\|$.
Letting $n \to +\infty$, from (5) one gets $\lambda\alpha\delta \geqslant (1-\lambda)\gamma$. Thus $\lambda \geqslant \dfrac{\gamma}{\gamma + \alpha\delta} > 0$.
On the other hand, for each $\varepsilon > 0$

$$C_F(y_n^*, \overline{x}) \leqslant C_F(y_n^*, x_n) + (\alpha + \varepsilon)\,\|x_n - \overline{x}\|.$$

Letting $n \to +\infty$, we get

$$C_F\left(\widetilde{y}^*, \overline{x}\right) \leqslant \liminf_{n\to+\infty} C_F(y_n^*, \overline{x}) \leqslant \lim_{n\to+\infty} -d(0, F(x_n)) = 0.$$

Since $0 \in F(\overline{x})$, we have $C_F(\widetilde{y}^*, \overline{x}) = 0$.
Finally

$$\begin{cases} 0 \in m^* \circ A_f(\overline{x}) - A_{C_F(y^*,\cdot)}(\overline{x}) \\ \qquad C_F(y^*, \overline{x}) = 0 \end{cases}$$

with $y^* = (1-\lambda)\lambda^{-1}\widetilde{y}^*$. □

REMARK 1. *The theorem abover remains true for a Pareto minimal solution* $\overline{x}$ *of* (P) *with respect to* M.

4. Application

In this section, we are concerned with the mathematical programming problem

$$(P^*)\min\ f(x) \qquad \text{subject to} \quad \begin{cases} g_i(x) \leqslant 0 & i = 1, 2, \ldots, m, \\ h_j(x) = 0 & j = 1, 2, \ldots, k, \end{cases}$$

where f, g_i, and h_j admit approximations at $\overline{x}$.

Setting $C := \left\{x \in X :\ g_i(x) \leqslant 0, h_j(x) = 0 \text{ for all } i, j\right\}$, $g(x) = (g_1(x), g_2(x), \ldots, g_m(x))$ and $h(x) = (h_1(x), h_2(x), \ldots, h_k(x))$, problem (P^*) is reduced to the problem (P), when the set-valued mapping F from X into $Y = \mathbb{R}^m \times \mathbb{R}^k$ is defined by

$$F(x) := (g(x), h(x)) + \mathbb{R}_+^m \times \{0_{\mathbb{R}^k}\};$$

here $\mathbb{R}^m_+$ is the nonnegative orthant of $\mathbb{R}^m$.

Obviously in that case, $Y^*_F = \mathbb{R}^m_+ \times \mathbb{R}^k$ and $M^* = \{1\}$ and for any $y^* = (\lambda, \mu) \in Y^*_F$ we have

$$C_F\left(y^*, x\right) =< \lambda, g\left(x\right) > + < \mu, h\left(x\right) > .$$

Take $\overline{x} \in C$ and $y^* = (\lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^k$, it can be verified that $C_F(y^*, \overline{x}) = 0$ if and only if $< \lambda, g(\overline{x}) >= 0$.

We deduce from Theorem 2 the following necessary condition for problem (P^*).

THEOREM 3. *Let $\overline{x}$ be a solution of (P^*). If (P^*) is regular at $\overline{x}$, then there exist vectors $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbb{R}^m_+$ and $\mu = (\mu_1, \mu_2, ..., \mu_m) \in \mathbb{R}^k$ such that*

$$\begin{cases} 0 \in A_f(\overline{x}) - \sum_{i=1}^m \lambda_i A g_i(\overline{x}) - \sum_{j=1}^k \mu_j A h_j(\overline{x}), \\ \qquad \lambda_i g_i(\overline{x}) = 0, \qquad i = 1, 2, ..., m. \end{cases}$$

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